Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(b) Impossible

Pf. If (x_n) converges and $(x_n + y_n)$ converges, then by the ALT, the sequence $\left(c_n = (x_n + y_n) - x_n = y_n \right)$ must also converge. Thus, it's impossible that (y_n) diverges. $\boxed{//}$

Tool Time Continues

Squeeze Thm.

Suppose $(a_n), (b_n), (c_n)$ are three sequences with $a_n \le b_n \le c_n$. Suppose that $\lim a_n$ & $\lim c_n$ exist.

Then $\lim b_n$ exists, and

$$\lim a_n \le \lim b_n \le \lim c_n.$$

<u>Eg:</u> Suppose we know that the function $\sin(x)$ is bounded between -1 & 1, i.e. $-1 \le \sin(x) \le 1$. What if we need to compute

$$\lim \frac{\sin(n^2+4n-3)n+1}{n^2}?$$

Observe that

$$\frac{(-1)n+1}{n^2} \le \frac{\sin(n^2+4n-3)n+1}{n^2} \le \frac{(1)n+1}{n^2}.$$

Observe that $\displaystyle \frac{(-1)n+1}{n^2} = (-1)\overbrace{\left(\frac{1}{n}\right)}^{\frac{n}{n^2}} + \overbrace{\left(\frac{1}{n}\right)^2}^{\frac{1}{n^2}}$

$\displaystyle \frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$

and $\displaystyle \frac{(1)n+1}{n^2} = \frac{1}{n} + \left(\frac{1}{n}\right)^2.$

We know $\lim \frac{1}{n} = 0$, and so by ALT,

(↑ we can assume.)

$$\lim \frac{(-1)n+1}{n^2} = (-1)\lim\left(\frac{1}{n}\right) + \lim\left(\frac{1}{n}\right)^2$$
$$= (-1)0 + 0^2 = 0.$$
$$\lim \frac{(1)n+1}{n^2} = (+1)\lim\left(\frac{1}{n}\right) + \lim\left(\frac{1}{n}\right)^2$$
$$= 1(0) + 0^2 = 0.$$

By the squeeze Thm,

$$\lim \frac{\sin(n^2+4n-3)n+1}{n^2} \text{ exists}$$

and is 0!

Monotone Convergence Theorem.

A monotone sequence is one that is increasing <u>or</u> is one that is decreasing.

That is, if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$, we say "(a_n) is increasing".

If $\forall n \in \mathbb{N}$, $b_{n+1} \leq b_n$,
we say that "(b_n) is decreasing".

<u>eg.</u> $(a_n) = (1, 2, 2, 2, 2, 2, \ldots)$
is increasing.
(and thus also monotone)

$(c_n) = (1.1, 4.1, 9.1, 16.1, \ldots)$
is increasing.

The sequence (d_n) defined by
$d_n = 2 - \frac{1}{n^2}$ is increasing.

<u>Pf:</u> $\forall n \in \mathbb{N}$, $d_{n+1} - d_n = 2 - \frac{1}{(n+1)^2} - \left(2 - \frac{1}{n^2}\right)$

$= \cancel{2} - \frac{1}{(n+1)^2} - \cancel{2} + \frac{1}{n^2}$

$= \frac{-n^2 + (n+1)^2}{n^2 (n+1)^2} = \frac{-n^2 + n^2 + 2n + 1}{n^2 (n+1)^2}$

$$= \frac{2n+1}{n^2(n+1)^2} > 0.$$

So $d_{n+1} - d_n > 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow d_{n+1} > d_n \quad \forall n \in \mathbb{N}$

$\Rightarrow (d_n)$ is <u>strictly increasing</u>.

$\boxtimes$

We say a sequence (a_n) is <u>bounded</u> if $\exists M > 0$ s.t.

$$|a_n| \le M \quad \forall n \in \mathbb{N}.$$

$$-M \le a_n \le M$$

<u>Note</u>: (a_n) is bounded $\Longleftrightarrow$ "S" $\{a_n : n \in \mathbb{N}\}$ is bounded below and bounded above.

(choose $M = \max\{|\sup S|, |\inf S|\}$.)

Thm (Monotone Convergence Thm). Every bounded monotone sequence converges.

Pf. Suppose (x_n) is an increasing sequence that is bounded. By AOC, the set $A = \{x_n : n \in \mathbb{N}\}$ has a sup, say $y = \sup A$. Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $y - x_N < \varepsilon$.

[If no such x_N existed, then $y - x_n \geq \varepsilon$ for all $n \in \mathbb{N}$, but then $y - \varepsilon \geq x_n$

$\forall n \in \mathbb{N}$, making $y-\varepsilon$ an upper bound for the set A. But that's impossible since $y = \sup A$.]

Note $y \geq x_N \Rightarrow y - x_N \geq 0$.

Also, $\forall n \geq N$, $x_n \geq x_N$

$\Rightarrow -x_n \leq -x_N$

and $y - x_n \leq y - x_N < \varepsilon$.

(Again $y - x_n \geq 0$ since $y \geq x_n$.)

So we have that $\forall n \geq N$,

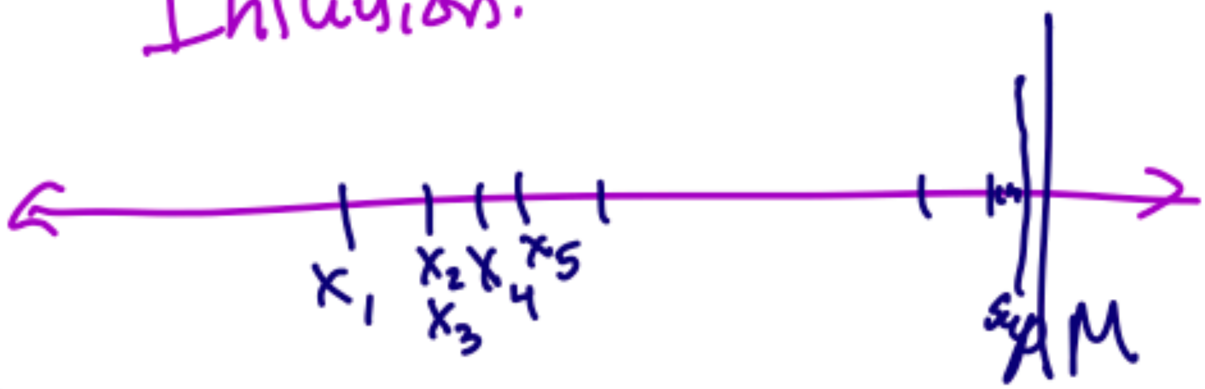
$$|y - x_n| = y - x_n < \varepsilon.$$

Thus, $\lim x_n = y$. ▨

(We would need a similar proof for the

case where (x_n) is decreasing
$x_n \to \inf A$.)

Intuition:



Example Let $x_1 = \sqrt{4}$

$$x_2 = \sqrt{4 + \sqrt{4}}$$
$$x_3 = \sqrt{4 + \underbrace{\sqrt{4+\sqrt{4}}}},$$

in general
$$x_{n+1} = \sqrt{4 + \underbrace{x_n}}$$

ⓐ Prove (x_n) converges.

ⓑ Find the limit.

Scratch

ⓐ Consider

$$x_{n+1}-x_n=\sqrt{4+x_n}-x_n$$
$$=\sqrt{4+x_n}-\sqrt{4+x_{n-1}}$$

$$x_{n+1}^2-x_n^2=4+x_n-x_n^2$$
$$=4+x_n-(4+x_{n-1})$$
$$=x_n-x_{n-1}$$

Need • show $x_n \geq 0$ th

② • Show $x_2 \geq x_1$

• Use induction and this algebra

✓ increasing.

① Bounded. $x_n \leq 10$

$0 \leq x_1 \leq 10$

induction

$$x_{n+1}=\sqrt{4+x_n} \leq \sqrt{4+10}$$
$$=\sqrt{14} \leq 10.$$

$$\left(x_1 = \sqrt{4} = 2,\ x_{n+1} = \sqrt{4 + x_n}\ \forall n \in \mathbb{N}. \right)$$

ⓐ (x_n) converges.

<u>Pf.</u> Observe that $0 \le 2 = \sqrt{4} = x_1 \le 10$.

Suppose that $0 \le x_n \le 10$ for some $n \in \mathbb{N}$.

Then $x_{n+1} = \sqrt{4 + x_n}$

$$\Rightarrow \quad 2 = \sqrt{4+0} \le \sqrt{4 + x_n} \le \sqrt{4+10} = \sqrt{14}.$$

Thus. $0 \le x_{n+1} \le 10$ as well.

By induction $0 \le x_n \le 10\ \forall n \in \mathbb{N}$.

Next, observe that $x_2 = \sqrt{4 + \sqrt{4}} = \sqrt{6}$

$$\ge x_1 = \sqrt{4} = 2.$$

Next, assume $\underline{x_{n+1} \ge x_n}$ for some

$n \in \mathbb{N}$. Then

$$x_{n+2}^2 - x_{n+1}^2 = \left(\sqrt{4 + x_{n+1}}\right)^2 - \left(\sqrt{4+x_n}\right)^2$$

$$= 4 + x_{n+1} - (4 + x_n)$$

$$= x_{n+1} - x_n \geq 0 \text{ by assumption.}$$

Thus, $x_{n+2}^2 - x_{n+1}^2 \geq 0$

$$\Rightarrow x_{n+2}^2 \geq x_{n+1}^2 \geq 0$$

$$\Rightarrow \underline{x_{n+2} \geq x_{n+1}} \geq 0.$$

By induction, $x_{n+1} \geq x_n \ \forall n \in \mathbb{N}$.

In summary, we have shown that (x_n) is a bounded sequence (i.e. $|x_n| \leq 10 \ \forall n \in \mathbb{N}$), and (x_n) is increasing.

By the MCT (monotone Convergence Thm)

(x_n) converges. $\blacksquare$